Modelling and Analysis of Fractional Order Systems using Ultradistributions *

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Abstract

In this paper we introduce a new mathematical tool to solve fractional equations representing models of fractional systems : The Ul-

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tradistributions.

Ultradistributions permit us to unify the notion of integral and derivative in one only operation. Several examples of application of the results obtained are given.

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1 Introduction

The use of fractional calculus for modelling physical systems has been considered in many works. See for example [1, 2, 3]. We can find also works dealing with the application of this mathematical tool in control theory [4, 5, 6, 7]..

Moreover, there are many physical systems that can be described by means of a fractional calculus. Some examples are: chaos [8], long electric lines [9], electrochemical process [10] and dielectric polarization [11].

In this paper we want to introduce a new mathematical framework to solve fractional equations representing models of fractional systems which was not treated in none of the previous works: The Ultradistributions.

The paper is organized as follow: in section 2 we introduce definition of fractional derivation and integration. In section 3 we give some examples of application of the formulae of section 2 using the Fourier Transform and the one-side Laplace Transform. In section 3 we present a circuital application. Finally in section 4 we discuss the results obtained in sections 1,2 and 3.

2 Fractional Calculus

The purpose of this sections is to introduce definition of fractional derivation and integration given in ref. [12]. This definition unifies the notion of integral and derivative in one only operation. Let $\hat{f}(x)$ a distribution of exponential type and $F(\Omega)$ the complex Fourier transformed Tempered Ultradistribution. Then:

$$F(\Omega) = U[\Im(\Omega)] \int\limits_{0}^{\infty} \hat{f}(x) e^{j\Omega x}\, dx - U[-\Im(\Omega)] \int\limits_{-\infty}^{0} \hat{f}(x) e^{j\Omega x}\, dx \tag{2.1}$$

($U(x)$ is the Heaviside step function) and

$$\hat{f}(x) = \frac{1}{2\pi} \oint\limits_{\Gamma} F(\Omega) e^{-j\Omega x}\, d\Omega \tag{2.2}$$

where the contour Γ surround all singularities of $F(\Omega)$ and runs parallel to real axis from $-\infty$ to ∞ above the real axis and from ∞ to $-\infty$ below the real axis. According to [12] the fractional derivative of $\hat{f}(x)$ is given by

$$\frac{d^\lambda \hat{f}(x)}{dx^\lambda} = \frac{1}{2\pi} \oint\limits_{\Gamma} (-j\Omega)^\lambda F(\Omega) e^{-j\Omega x}\, d\Omega + \oint\limits_{\Gamma} (-j\Omega)^\lambda a(\Omega) e^{-j\Omega x}\, d\Omega \tag{2.3}$$

Where $a(\Omega)$ is entire analytic and rapidly decreasing. If $\lambda = -1$, d^λ/dx^λ is the inverse of the derivative (an integration). In this case the second term of the right side of (2.3) gives a primitive of $\hat{f}(x)$. Using Cauchy's theorem the

additional term is

$$\oint \frac{a(\Omega)}{\Omega} e^{-j\Omega x} d\Omega = 2\pi a(0) \tag{2.4}$$

Of course, an integration should give a primitive plus an arbitrary constant. Analogously when $\lambda = -2$ (a double iterated integration) we have

$$\oint \frac{a(\Omega)}{\Omega^2} e^{-j\Omega x} d\Omega = \gamma + \delta x \tag{2.5}$$

where γ and δ are arbitrary constants. With the change of variables $s = -j\Omega$ formulae (2.1) and (2.2) can be writen as:

$$G(s) = U[\Re(s)] \int_0^{\infty} \hat{f}(x) e^{-sx}\, dx - U[-\Re(s)] \int_{-\infty}^{0} \hat{f}(x) e^{-sx}\, dx \tag{2.6}$$

and

$$\hat{f}(x) = \frac{1}{2\pi i} \oint_{\Gamma} G(s) e^{sx}\, ds \tag{2.7}$$

where the contour Γ surround all singularities of $G(S)$ and runs parallel to imaginary axis from $-j\infty$ to $j\infty$ to the right of the imaginary axis and from $j\infty$ to $-j\infty$ to the left of the imaginary axis. Formula (2.6) represents the two-sided Lapnace Transform. The fractional derivative is now:

$$\frac{d^{\lambda}\hat{f}(x)}{dx^{\lambda}} = \frac{1}{2\pi i} \oint_{\Gamma} s^{\lambda} G(s) e^{sx}\, ds + \oint_{\Gamma} s^{\lambda} a(s) e^{sx}\, ds \tag{2.8}$$

For the one-side Laplace Transform we have

$$G(s) = U[\Re(s)] \int_0^{\infty} \hat{f}(x) e^{-sx}\, dx \tag{2.9}$$

$$\hat{f}(x) = \frac{1}{2\pi j} \int\limits_{a-j\infty}^{a+j\infty} G(s)e^{sx}\, ds \tag{2.10}$$

and for the fractional derivative:

$$\frac{d^{\lambda}\hat{f}(x)}{dx^{\lambda}} = \frac{1}{2\pi j} \int\limits_{a-j\infty}^{a+j\infty} s^{\lambda}G(s)e^{sx}\, ds \tag{2.11}$$

3 Examples

In this section we give some examples of the application of formulae of the precedent section. At first using the Fourier Transform and at second place using the one-side Laplace Transform.

The Fourier Transform

Let $U(x)$ be the Heaviside step function.

$$\hat{f}(x) = U(x) \quad ; \quad F(\Omega) = U[\Im(\Omega)] \int\limits_{0}^{\infty} e^{-j\Omega x}\, dx = \frac{jU[\Im(\Omega)]}{\Omega} \tag{3.1}$$

The fractional derivative is:

$$\frac{d^{\lambda}U(x)}{dx^{\lambda}} = \frac{je^{-\frac{j\pi\lambda}{2}}}{2\pi} \oint\limits_{\Gamma} U[\Im(\Omega)]\Omega^{\lambda-1}e^{-j\Omega x}\, d\Omega + \oint\limits_{\Gamma} \Omega^{\lambda}a(\Omega)e^{-j\Omega x}\, d\Omega =$$

$$\frac{je^{\frac{-j\pi\lambda}{2}}}{2\pi} \int\limits_{-\infty}^{\infty} (\omega + j0)^{\lambda-1}e^{-j\omega x}\, d\omega + \oint\limits_{\Gamma} \Omega^{\lambda}a(\Omega)e^{-j\Omega x}\, d\Omega \tag{3.2}$$

With the use of the result (see ref.[13])

$$\int_{-\infty}^{\infty} (\omega + j0)^{\lambda-1} e^{-j\omega x} \, d\omega = -2\pi j \frac{e^{\frac{i\pi\lambda}{2}}}{\Gamma(1-\lambda)} x_+^{-\lambda} \tag{3.3}$$

we obtain:

$$\frac{d^\lambda U(x)}{dx^\lambda} = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} + \oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x} \, d\Omega \tag{3.4}$$

When $\lambda = n$

$$\left. \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} \right|_{\lambda=n} = \delta^{(n-1)}(x) \tag{3.5}$$

$$\oint_\Gamma \Omega^n a(\Omega) e^{-j\Omega x} \, d\Omega = 0 \tag{3.6}$$

and we have the ordinary derivative:

$$\frac{d^n U(x)}{dx^n} = \delta^{(n-1)}(x) \tag{3.7}$$

When $\lambda = -n$

$$\frac{d^{-n} U(x)}{dx^{-n}} = \frac{x_+^n}{n!} + a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{3.8}$$

which is a n-times iterated integral.

Let $\delta(x)$ the Dirac's delta distribution. For it we have:

$$\hat{f}(x) = \delta(x) \quad ; \quad F(\Omega) = \frac{Sgn[\Im(\Omega)]}{2} \tag{3.9}$$

The fractional derivative is:

$$\frac{d^\lambda \delta(x)}{dx^\lambda} = \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} + \oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x} \, d\Omega \tag{3.10}$$

When $\lambda = n$:

$$\frac{d^n \delta(x)}{dx^n} = \delta^{(n)}(x) \tag{3.11}$$

and when $\lambda = -n$:

$$\frac{d^{-n} \delta(x)}{dx^{-n}} = \frac{x_+^{n-1}}{(n-1)!} + a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{3.12}$$

Let us consider now the fractional derivative of e^{jbx}

$$\hat{f}(x) = e^{jbx} \quad ; \quad F(\Omega) = \frac{j}{\Omega + b} \tag{3.13}$$

We have:

$$\frac{d^\lambda e^{jbx}}{dx^\lambda} = \frac{j}{2\pi} \oint_\Gamma \frac{(-j\Omega)^\lambda e^{-j\Omega x}}{\Omega + b} \, d\Omega + \oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x} \, d\Omega = \tag{3.14}$$

$$\frac{ie^{\frac{-i\pi\lambda}{2}}}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega + j0)^\lambda}{\omega + b + j0} e^{-j\omega x} d\omega - \frac{ie^{\frac{-i\pi\lambda}{2}}}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega - j0)^\lambda}{\omega + b - j0} e^{-j\omega x} d\omega +$$

$$\oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x} \, d\Omega \tag{3.15}$$

From ref.[14] we obtain:

$$\int_{-\infty}^{\infty} \frac{(x + \gamma)^\lambda}{x + \beta} e^{-ipx} dx =$$

$$2\pi U(p) \frac{e^{\frac{-j\pi}{2}(1-\lambda)}}{\Gamma(1-\lambda)} p^{-\lambda} e^{i\beta p} \phi[-\lambda, 1-\lambda, j(\gamma - \beta)p] \tag{3.16}$$

where ϕ is the confluent hypergeometric function. Thus the fractional derivative is:

$$\frac{d^\lambda e^{jbx}}{dx^\lambda} = \frac{(x+j0)^{-\lambda}}{\Gamma(1-\lambda)}\phi(1,1-\lambda,jbx) + \oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x}\, d\Omega \tag{3.17}$$

With the use of equality:

$$\phi(1,1-\lambda,jbx) = (jbx)^\lambda e^{jbx}\left[\Gamma(1-\lambda) + \lambda\Gamma(-\lambda,jbx)\right] \tag{3.18}$$

where $\Gamma(z_1,z_2)$ is the incomplete gamma function, (3.17) takes the form:

$$\frac{d^\lambda e^{jbx}}{dx^\lambda} = (jb)^\lambda e^{jbx}\left[1 + \frac{\lambda}{\Gamma(1-\lambda)}\Gamma(-\lambda,jbx)\right] + \oint_\Gamma \Omega^\lambda a(\Omega) e^{-j\Omega x}\, d\Omega \tag{3.19}$$

When $\lambda = n$

$$\frac{d^n e^{jbx}}{dx^n} = (jb)^n e^{jbx} \tag{3.20}$$

and when $\lambda = -n$:

$$\frac{d^{-n} e^{jbx}}{dx^{-n}} = (jb)^{-n} e^{jbx} + a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \tag{3.21}$$

The Laplace Transform

If we use the one-side Laplace transform to evaluate the fractional derivative of $U(x)$,then:

$$\hat{f}(x) = U(x) \quad ; \quad G(s) = U[\Re(s)]\int_0^\infty e^{-sx}dx = \frac{U[\Re(s)]}{s} \tag{3.22}$$

and as a consequence:

$$\frac{d^\lambda U(x)}{dx^\lambda} = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} U[\Re(s)]s^{\lambda-1}e^{sx}\,ds = \tag{3.23}$$

$$\frac{e^{-ax}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{jsx}}{(a+js)^{1-\lambda}}ds = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} \tag{3.24}$$

$$\frac{d^\lambda U(x)}{dx^\lambda} = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} \tag{3.25}$$

When $\lambda = n$ we obtain

$$\frac{d^n U(x)}{dx^n} = \delta^{(n-1)}(x) \tag{3.26}$$

which coincides with (3.7). When $\lambda = -n$ the result is:

$$\frac{d^{-n} U(x)}{dx^{-n}} = \frac{x_+^n}{n!} \tag{3.27}$$

In a analog way we obtain for Dirac's delta distribution:

$$\frac{d^\lambda \delta(x)}{dx^\lambda} = \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} \tag{3.28}$$

$$\frac{d^n \delta(x)}{dx^n} = \delta^{(n)}(x) \tag{3.29}$$

$$\frac{d^{-n} \delta(x)}{dx^{-n}} = \frac{x_+^{n-1}}{(n-1)!} \tag{3.30}$$

Finally we consuder the fractional derivative of e^{jbx}:

$$\hat{f}(x) = U(x)e^{jbx} \quad ; \quad G(s) = \frac{U[\Re(s)]}{s-ib} \tag{3.31}$$

According to (2.11):

$$\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = \frac{1}{2\pi j}\int\limits_{a-j\infty}^{a+j\infty}\frac{U[\Re(s)]}{s-jb}s^\lambda e^{sx}ds = \tag{3.32}$$

$$-\frac{e^{-\frac{j\pi\lambda}{2}}}{2\pi j}\int\limits_{-\infty}^{\infty}\frac{(s+j0)^\lambda}{s+b+j0}e^{-jsx}ds \tag{3.33}$$

And thus:

$$\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = \frac{U(x)x^{-\lambda}}{\Gamma(1-\lambda)}\phi(1, 1-\lambda, jbx) \tag{3.34}$$

Using (3.18), (3.34) transforms into:

$$\frac{d^\lambda U(x)e^{jbx}}{dx^\lambda} = (jb)^\lambda U(x)e^{jbx}\left[1+\frac{\lambda}{\Gamma(1-\lambda)}\Gamma(-\lambda, jbx)\right] \tag{3.35}$$

When $\lambda = n$:

$$\frac{d^n e^{jbx}}{dx^n} = (jb)^n U(x)e^{jbx} \tag{3.36}$$

and when $\lambda = -n$:

$$\frac{d^{-n} e^{jbx}}{dx^{-n}} = (jb)^{-n} U(x)e^{jbx} \tag{3.37}$$

4 Circuital Application

As circuital application we consider a semi-infinite cable with a voltage $V = V_0 e^{j\omega t}$ applied at one end. We use first the Fourier transform and then the Laplace transform for see the diferences between both treatments.

The Fourier Transform

We should solve the system:

$$\begin{cases} \frac{\partial^2 f(x,t)}{\partial x^2} - RC\frac{\partial f(x,t)}{\partial t} = 0 \quad ; \quad x > 0 \\ \\ f(0,t) = V_0 e^{j\omega t} \end{cases} \tag{4.1}$$

where R is the resistance per unit length and C is the capacitance per unit length. Let $V(x,t)$ the voltage along the semi-infinite cable. We use a formalism developed in ref.[15] to solve the system (4.1). It consist in to define:

$$\begin{cases} V(x,t) = U(x)f(x,t) \\ \\ g(t) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=0} \end{cases} \tag{4.2}$$

The differential equation in (4.1) transforms into:

$$\frac{\partial^2 V(x,t)}{\partial x^2} - RC\frac{\partial V(x,t)}{\partial t} = \delta^{'}(x)V_0 e^{j\omega t} + \delta(x)g(t) \tag{4.3}$$

Taking the Fourier transform of (4.3) we obtain:

$$\hat{V}(\alpha_1, \alpha_2) = \mathcal{F}[V(x,t)] \tag{4.4}$$

$$\hat{V}(\alpha_1, \alpha_2) = \pi j V_0 \delta(\alpha_1 + \omega)\left[\frac{1}{\alpha_2 - \frac{1-j}{\sqrt{2}}\sqrt{-\alpha_1 RC}} + \frac{1}{\alpha_2 + \frac{1-j}{\sqrt{2}}\sqrt{-\alpha_1 RC}}\right] - \frac{\hat{g}(\alpha_1)}{(1-j)\sqrt{-2\alpha_1 RC}}$$

$$\left[\frac{1}{\alpha_2 - \frac{1-j}{\sqrt{2}}\sqrt{-\alpha_1 RC}} - \frac{1}{\alpha_2 + \frac{1-j}{\sqrt{2}}\sqrt{-\alpha_1 RC}}\right] \tag{4.5}$$

Deprecating the exponential increasing in the solution we obtain:

$$\hat{g}(\alpha_1) = -(1+j)\pi\sqrt{-2\alpha_1 RC}\;\delta(\alpha_1 + \omega) \tag{4.6}$$

and then we obtain:

$$V(x,t) = V_0 U(x) e^{-\sqrt{\frac{\omega RC}{2}}x} e^{j(\omega t - \sqrt{\frac{\omega RC}{2}}x)} \tag{4.7}$$

$$g(t) = -(1+j)\sqrt{\frac{\omega RC}{2}}\;V_0 e^{j\omega t} \tag{4.8}$$

The current $i(x,t)$ is:

$$i(x,t) = -\frac{1}{R}\frac{\partial V(x,t)}{\partial x} \quad ; \quad x > 0 \tag{4.9}$$

As:

$$\frac{\partial V(x,t)}{\partial x} = (1+j)\sqrt{\frac{\omega RC}{2}} V_0 e^{-\sqrt{\frac{\omega RC}{2}}x} e^{j(\omega t - \sqrt{\frac{\omega RC}{2}}x)}\;;\; x > 0 \tag{4.10}$$

then:

$$i(x,t) = (1+j)\sqrt{\frac{\omega C}{2R}} V_0 e^{-\sqrt{\frac{\omega RC}{2}}x} e^{j(\omega t - \sqrt{\frac{\omega RC}{2}}x)}\;;\; x > 0 \tag{4.11}$$

If we take $\lambda = 1/2$ in (3.19 we obtain:

$$\frac{d^{\frac{1}{2}} e^{j\omega t}}{dt^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}} e^{j\omega t}\left[1 + \frac{1}{2\sqrt{\pi}}\Gamma(-\frac{1}{2}, j\omega t)\right] + \oint_{\Gamma} Z^{\frac{1}{2}} a(Z) e^{-jZt} dZ \tag{4.12}$$

$$\frac{\partial^{\frac{1}{2}}V(x,t)}{\partial t^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}}\left[1+\frac{1}{2\sqrt{\pi}}\Gamma(-\frac{1}{2},j\omega t)\right]e^{-\sqrt{\frac{\omega RC}{2}}x}e^{j(\omega t-\sqrt{\frac{\omega RC}{2}}x)}+ \oint_{\Gamma} Z^{\frac{1}{2}}a(Z,x)e^{-jZt}dZ \tag{4.13}$$

Thus we have a relation between the current and the time derivative of the voltage:

$$i(x,t) = \sqrt{\frac{C}{R}}\left\{\left[\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{(j\omega)^{\frac{1}{2}}\Gamma(-\frac{1}{2},j\omega t)}{2\sqrt{\pi}}\right]V(x,t) - \oint_{\Gamma} Z^{\frac{1}{2}}a(Z,x)e^{-jZt}dZ\right\} \tag{4.14}$$

If we consider only the first term in the rigth side of (4.14) we obtain the more habitual result:

$$i(x,t) = \sqrt{\frac{C}{R}}\frac{\partial^{\frac{1}{2}}V(x,t)}{\partial t^{\frac{1}{2}}} \tag{4.15}$$

The Laplace Transform

If we use the Laplace transform in place of the Fourier transform to evaluate the fractional derivatives, (4.12),(4.13) and (4.14) are replaced by:

$$\frac{d^{\frac{1}{2}}e^{j\omega t}}{dt^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}}e^{j\omega t}\left[1+\frac{1}{2\sqrt{\pi}}\Gamma(-\frac{1}{2},j\omega t)\right] \tag{4.16}$$

$$\frac{\partial^{\frac{1}{2}}V(x,t)}{\partial t^{\frac{1}{2}}} = (j\omega)^{\frac{1}{2}}\left[1+\frac{1}{2\sqrt{\pi}}\Gamma(-\frac{1}{2},j\omega t)\right]e^{-\sqrt{\frac{\omega RC}{2}}x}e^{j(\omega t-\sqrt{\frac{\omega RC}{2}}x)} \tag{4.17}$$

$$i(x,t) = \sqrt{\frac{C}{R}}\left[\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} - \frac{(j\omega)^{\frac{1}{2}}\Gamma(-\frac{1}{2},j\omega t)}{2\sqrt{\pi}}\right]V(x,t) \tag{4.18}$$

Difference between this results and the precedents is the term that contain a contour integral.

5 Discussion

In this paper we have shown that Ultradistribution Theory is an adequate framework to define a Fractional Caculus and its applications. This definition unifies the notion of integral and derivative in one only operation. Several examples of application of fractional derivative are given, including a circuital application: a semi-infinite cable with a voltage $V = V_0 e^{j\omega t}$ applied at one end.

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